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Tensor products of perfect modules and maximal surjective Buchsbaum modules

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Abstract

In this note, we investigate tensor products of perfect modules and maximal surjective Buchsbaum modules over any CM local ring. In particular, we should prove that the above tensor products are always Buchsbaum modules. © 1998 Elsevier Science B.V.

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1. Introduction

The main purpose of this note is to prove the following theorem.

Theorem 1.1. *Let A be a CM local ring, M a perfect A-module, that is,* $\operatorname{grade}_A M = \operatorname{pd}_A M$ *and N a maximal surjective Buchsbaum A-module. Then*

(1) *$M \otimes_A N$ is always a Buchsbaum A-module.*

(2) $\dim M \otimes_A N = \dim M$.

(3) $\operatorname{depth} M \otimes_A N = \max\{\operatorname{depth} N - \operatorname{pd}_A M, 0\}$.

In particular, when $\operatorname{depth} N \geq \operatorname{pd}_A M$,

(4) *$M \otimes_A N$ is a surjective Buchsbaum A-module.*

Corollary 1.2. *Let A be a CM local ring and M a perfect A-module with* $\dim M \geq 1$. *Suppose that N is a maximal surjective Buchsbaum A-module. Then the following conditions are equivalent:*

(1) *N is a maximal CM A-module.*

(2) *$M \otimes_A N$ is a CM A-module.*

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We first recall several definitions. Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and the residue field $k = A/\mathfrak{m}$. Let M be a finite A-module.

The grade of M, the projective dimension of M, the injective dimension of M, the minimal number of generators and the length of M are denoted by $\mathrm{grade}_A M, \mathrm{pd}_A M, \mathrm{id}_A M, \mu_A(M)$ and $\ell_A(M)$, respectively (cf. e.g. [6]). Moreover, for any parameter ideal J of M, the multiplicity of M with respect to J is denoted by $e_J(M)$.

We put $\mu_A^i(M) = \dim_k \mathrm{Ext}_A^i(k,M)$, the ith Bass number of M and put $\beta_i^A(M) = \dim_k \mathrm{Tor}_i^A(k,M)$, the ith Betti number of M.

M is called an *F.L.C.* if the local cohomology modules $H_{\mathfrak{m}}^i(M)$ are of finite length for all $i < r = \dim M$. Then we put

$$I_A(M) = \sum_{i=0}^{r-1} \binom{r-1}{i} \ell_A\left(H_{\mathfrak{m}}^i(M)\right).$$

M is called a *Buchsbaum* A-module if the difference $\ell_A(M/JM) - e_J(M)$ is independent of J for all parameter ideals J of M. Then we have $I_A(M) = \ell_A(M/JM) - e_J(M)$ for all above J. See [10, 11] for details.

M is called a *surjective Buchsbaum* A-module if the natural map $\varphi_M^i : \mathrm{Ext}_A^i(k,M) \to H_{\mathfrak{m}}^i(M)$ is surjective for all $i < \dim M$. See [5, 7, 12] for details. For example, all CM A-modules and all linear maximal Buchsbaum A-modules are surjective Buchsbaum A-modules (cf. [13]). On the other hand, in general, any surjective Buchsbaum A-module is a Buchsbaum A-module. When A is regular, the converse is true; see [10, Theorem 2.10, Corollary 2.16]. Moreover, M is called maximal if $\dim M = \dim A$.

In [5], Kawasaki proved the following theorem.

Theorem 1.3 (Kawasaki [5, Theorem 3.3]). *Let A be a CM local ring and K_A a canonical module of A. Let M be a finite A-module of finite projective dimension. Then the following statements hold*:

(1) *$M \otimes_A K_A$ is CM if and only if so is M.*

(2) *If $M \otimes_A K_A$ is a surjective Buchsbaum A-module, then so is M.*

Remark 1. There exists a surjective Buchsbaum A-module M of finite projective dimension such that $M \otimes_A K_A$ is not surjective Buchsbaum (cf. [5, Proposition 4.1]).

In this note, we shall improve the statement (1) of the above theorem and investigate the structure of tensor products of perfect modules and maximal surjective Buchsbaum modules over any CM local ring; see Theorem 1.1.

In order to prove Theorem 1.1, we raise the following well-known facts (cf. e.g. [9]).

Let A be a local ring and M, N are finite A-modules. Then:

1.4. $\mathrm{depth}\, A \le \mathrm{grade}_A M + \dim M \le \dim A$.

1.5. $\mathrm{grade}_A M \le \mathrm{pd}_A M$.

1.6. *Auslander–Buchsbaum*: When $\mathrm{pd}_A M < \infty$, one has $\operatorname{depth} A = \mathrm{pd}_A M + \operatorname{depth} M$.

1.7. *Intersection Theorem*: When $\mathrm{pd}_A M < \infty$, one has $\dim N \leq \mathrm{pd}_A M + \dim M \otimes_A N$.

1.8. If A admits a CM A-module of finite projective dimension, then it is a CM local ring; see Proposition 4.1.

1.9. If A is a CM local ring and $\mathrm{pd}_A M < \infty$, then M is perfect if and only if M is CM.

In Section 2, we shall prove Theorem 1.1 and Corollary 1.2. The proof of Theorem 1.1 consists of three pieces as follows:

First, in Section 2.1, we prove that the tensor products of perfect modules and maximal CM modules are CM modules.

Next, in the Section 2.2, we calculate the local cohomology modules of the tensor product $M \otimes_A N$; see Proposition 2.7.

Finally, in the Section 2.3, we complete the proof of Theorem 1.1, using finite injective hulls.

In Section 3, we give several examples with respect to Theorem 1.1. For example, the tensor products of any typical (cf. [5]) surjective Buchsbaum A-module and maximal CM A-module are surjective Buchsbaum A-modules over any complete CM local ring A; see Proposition 3.6. On the other hand, we give many examples of the tensor product of typical surjective Buchsbaum A-module and maximal surjective A-module which is not a surjective Buchsbaum A-module; see Example 3.7.

In Section 4, we prove the following result.

Proposition 1.4. *Let A be a local ring and n a nonnegative integer. If A admits a finite A-module M of finite projective dimension which satisfies the Serre condition S_n, then A itself satisfies S_n.*

2. Proof of Theorem 1.1

2.1. Maximal CM modules

Let A be a local ring and M, N nonzero finite A-modules. We first give several lemmas.

Lemma 2.1. *Suppose that M is perfect and* $\dim N = \dim A = d$. *Then* $\dim M \otimes_A N = \dim M$.

Proof. The inequality $\dim M \otimes_A N \leq \dim M$ is clear.

On the other hand, by 1.7 and 1.4, one has

$$\dim M \otimes_A N \geq \dim N - \mathrm{pd}_A M = d - \mathrm{grade}_A M \geq \dim M.$$

Hence $\dim M \otimes_A N = \dim M$. □

Lemma 2.2. *Suppose that* $\mathrm{pd}_A M < \infty$ *and* N *is a maximal CM A-module. Then* $\mathrm{Tor}_i^A(M,N) = 0$ *for all* $i \geq 1$.

Proof. Let $\mathbb{F} : 0 \to F_s \xrightarrow{\phi_s} F_{s-1} \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 \to 0$ be a minimal free resolution of M over A, where $s = \mathrm{pd}_A M$. From the Buchsbaum–Eisenbud criterion [2], we have $\mathrm{grade}(I_{r_i}(\phi_i), A) \geq i$ for all $i \geq 1$, where $r_i = \sum_{j=i}^{s}(-1)^j \mathrm{rank}_A F_j$.

Since N is a maximal CM A-module, we get $\mathrm{grade}(I_{r_i}(\phi_i), N) \geq i$ for all $i \geq 1$. Therefore $\mathbb{F} \otimes_A N$ is acyclic, that is, $\mathrm{Tor}_i^A(M,N) = 0$ for all $i \geq 1$. □

Lemma 2.3 (Strooker [9, Theorem 7.1.2]). *Suppose that* $s = \mathrm{pd}_A M < \infty$ *and* $\mathrm{Tor}_i^A(M,N) = 0$ *for all* $i \geq 1$. *Then*

$$\mu_A^i(M \otimes_A N) = \sum_{p=0}^{s} \beta_p^A(M)\, \mu_A^{i+p}(N) \quad \text{for all } i.$$

From these lemmas, we obtain that the following proposition.

Proposition 2.4. *Suppose that M is a perfect A-module and N is a maxiaml CM A-module. Then* $M \otimes_A N$ *is a CM A-module with* $\dim M \otimes_A N = \dim M$.

Proof. Put $d = \dim A$ and $s = \mathrm{pd}_A M < \infty$. By Lemmas 2.2 and 2.3, one can easily get $\mathrm{depth}\, M \otimes_A N = d - s$. In fact, $\mu_A^i(M \otimes_A N) = 0$ for all $i < d-s$ and $\mu_A^{d-s}(M \otimes_A N) \geq \beta_s^A(M)\mu_A^d(N) \geq 1$.

Now suppose that M is perfect. Then $\dim M \otimes_A N = \dim M \leq \dim A - \mathrm{grade}_A M = d - s$. Hence $M \otimes_A N$ is a CM A-module as required. □

Now suppose that $\mathrm{pd}_A M < \infty$. By the above argument, we have

$$\mathrm{depth}\, M \otimes_A N = \dim A - \mathrm{pd}_A M$$

for any maximal CM module N. Furthermore, assume that $\dim A/I + \mathrm{ht}_A I = \dim A$ for any ideal I of A. Then we get

$$\dim M = \dim A - \mathrm{ht}(\mathrm{ann}_A M) = \dim A - \mathrm{grade}_A M,$$

where the last equality follows from [9, Corollary 9.1.6].

Hence we get the following result. This improves Theorem 1.3(i).

Corollary 2.5. *Suppose that* $\mathrm{pd}_A M < \infty$ *and* $\dim A/I + \mathrm{ht}_A I = \dim A$ *for any ideal* I *of* A. *Then the following conditions are equivalent*:

(1) *M is perfect.*

(2) $M \otimes_A N$ *is a CM A-module with* $\dim M \otimes_A N = \dim M$ *for any maximal CM A-module N.*

(3) $M \otimes_A N$ *is a CM A-module with* $\dim M \otimes_A N = \dim M$ *for some maximal CM A-module N.*

We now recall the following conjecture by Auslander.

Conjecture 2.6. (*Codimension Conjecture*). *When* $\mathrm{pd}_A M < \infty$, *one has*

$$\dim M + \mathrm{grade}_A M = \dim A.$$

Remark 2. If Codimension Conjecture is true, Corollary 2.5 is true for any local ring.

2.2. *Modules of finite injective dimension*

Unless specified, throughout of this subsection, let A be a CM local ring of dimension d, M a perfect A-module and N a maximal surjective Buchsbaum A-module.

The aim of this subsection is to state and to prove the following proposition.

Proposition 2.7. *Let A, M and N be as above and suppose* $\mathrm{id}_A N < \infty$. *Then* $M \otimes_A N$ *is a Buchsbaum A-module with* $\dim M \otimes_A N = \dim M$. *Furthermore, we get*

$$H^j_{\mathfrak{m}}(M \otimes_A N) \cong \bigoplus_{i=0}^{d-1} \mathrm{Tor}^A_{i-j}(M,k)^{h^i_A(N)} \quad \textit{for all } j < \dim M,$$

where $h^i_A(N) = \ell(H^i_{\mathfrak{m}}(N))$ *for all* $i < d$.

In order to prove this proposition, we need the structure theorem which was proved by Goto and Kawasaki (cf. [3, 5]).

Theorem 2.8 (Goto [3] and Kawasaki [5, Theorem 3.1]). *Let A be a complete CM local ring of dimension d and K_A a canonical module of A. Let $\mathbb{F} \to k$ be a minimal free resolution of k over A. Let $(-)^*$ denote the functor* $\mathrm{Hom}_A(-,A)$. *We will define L_i as follows:*

$$0 \to F_0^* \otimes K_A \to F_1^* \otimes K_A \to \cdots \to F_{d-i}^* \otimes K_A \to L_i \to 0 \quad (\mathrm{ex})$$

for $i = 0, 1, \ldots, d$. Then L_i is an indecomposable maximal (except L_0 in the case A is regular) surjective Buchsbaum A-module of finite injective dimension such that

$$h^i_A(L_j) = \delta_{ij} \quad \textit{for all } i,j = 0,1,\ldots,d,$$

where $h^d_A(N)$ denotes $\ell(\mathrm{Im}\,\varphi^d_N)$ for any finite A-module N with $\dim N = d$.

Note that we have $L_0 = k$ provided that A is regular.

Furthermore, for any maximal surjective Buchsbaum A-module N with $\mathrm{id}_A N < \infty$, *we can write it as follows:*

$$N \cong \bigoplus_{i=0}^{d} L_i^{h^i_A(N)}.$$

On the other hand, considering a mapping cone of a minimal free resolution, we get the following lemma.

Lemma 2.9. *Let A be a local ring and M a finite A-module. Then for each i, $\ell(\mathrm{Tor}_i^A(M/xM,k))$ is independent of the choice of the M-regular element x.*

Proof of Proposition 2.7. From Lemma 2.1, $\dim M \otimes_A N = \dim M = r$. We may assume that A is complete and $r \geq 1$. By Theorem 2.8, it is enough to show that $M \otimes_A L_i$ is a Buchsbaum A-module for $i = 0, 1, \ldots, d$.

Step 1: In the case of $i = 0$, we assume that A is not regular and put $\overline{L_0} = L_0/H^0_{\mathfrak{m}}(L_0)$. Then as $\overline{L_0}$ is a maximal CM A-module, we get by Lemma 2.2 as follows:

$$0 = \mathrm{Tor}_1^A(M,\overline{L_0}) \to M \otimes_A k \to M \otimes_A L_0 \to M \otimes_A \overline{L_0} \to 0 \quad (\mathrm{ex}).$$

Since $M \otimes_A \overline{L_0}$ is a CM A-module, we have $M \otimes_A L_0$ is a surjective Buchsbaum A-module with

$$H^0_m(M \otimes L_0) = M \otimes_A k, \qquad H^i_m(M \otimes L_0) = 0 \quad \text{for all } 1 \leq i < r.$$

Note that the above formula also holds for any regular local ring A.

Step 2: In the case of $i = 1$, put $X_k = \mathrm{Hom}_A(\mathrm{syz}^A_d(k), K_A)$. Then $M \otimes_A X_k$ is a CM A-module.

Moreover, from the short exact sequence

$$0 \to L_1 \to X_k \to k \to 0 \quad (\mathrm{ex}),$$

and from Lemma 2.2, we get

$$0 \to \mathrm{Tor}_1^A(M,k) \to M \otimes_A L_1 \to M \otimes_A X_k \to M \otimes_A k \to 0 \quad (\mathrm{ex}).$$

Let J be any parameter ideal for $M \otimes_A L_1$. Then it is one for M, because of $\mathrm{Supp}_A(L_1) = \mathrm{Spec}(A)$. Hence the above exact sequence implies that

$$e_J(M \otimes_A L_1) = e_J(M \otimes_A X_k).$$

On the other hand, as $\mathrm{pd}_A M/JM < \infty$, we get

$$0 \to \mathrm{Tor}_1^A(M/JM,k) \to \frac{M \otimes L_1}{J(M \otimes L_1)} \to \frac{M \otimes X_k}{J(M \otimes X_k)} \to M \otimes_A k \to 0 \quad (\mathrm{ex}).$$

From here, we get

$$\ell\left(\frac{M \otimes L_1}{J(M \otimes L_1)}\right) - e_J(M \otimes_A L_1) = \ell(\mathrm{Tor}_1^A(M/JM,k)) - \ell(M/\mathfrak{m}M).$$

Since M is a CM A-module, this is independent of the choice of the parameter ideal J for M; see Lemma 2.9. Therefore $M \otimes_A L_1$ is a Buchsbaum A-module.

Now put $N = \ker(M \otimes_A X_k \to M \otimes_A k)$, and one can get the following two short exact sequences:

$$0 \to \mathrm{Tor}_1^A(M,k) \to M \otimes_A L_1 \to N \to 0 \quad (\mathrm{ex}),$$

$$0 \to N \to M \otimes_A X_k \to M \otimes_A k \to 0 \quad (\mathrm{ex}).$$

Since $\ell(\mathrm{Tor}_1^A(M,k)) < \infty$ and depth $N > 0$, we have

$$H^0_{\mathfrak{m}}(M \otimes_A L_1) \cong \mathrm{Tor}_1^A(M,k), \qquad H^j_{\mathfrak{m}}(M \otimes_A L_1) \cong H^j_{\mathfrak{m}}(N) \quad (\forall j \geq 1).$$

Moreover, as $M \otimes_A X_k$ is CM, we get

$$H^j_{\mathfrak{m}}(N) \cong H^{j-1}_{\mathfrak{m}}(M \otimes_A k) \cong \mathrm{Tor}^A_{1-j}(M,k)$$

for all $1 \leq j < \dim M = r$. Hence the assertion is true for $i = 1$.

Step 3: Assume that $M \otimes_A L_{i-1}$ is a Buchsbaum A-module for some i $(2 \leq i \leq d)$ and

$$H^j_{\mathfrak{m}}(M \otimes_A L_{i-1}) \cong \mathrm{Tor}^A_{i-j-1}(M,k) \quad \text{for } j < r.$$

In order to complete the proof, we have to show that $M \otimes_A L_i$ is a Buchsbaum A-module and

$$H^j_{\mathfrak{m}}(M \otimes_A L_i) \cong \mathrm{Tor}^A_{i-j}(M,k) \quad \text{for } j < r.$$

In order to do that, we consider the following exact sequence:

$$0 \to L_i \to F^*_{d-i+1} \otimes_A K_A \to L_{i-1} \to 0 \quad (\text{ex}).$$

For simplicity, we put $X = F^*_{d-i+1} \otimes_A K_A$. Applying the functor $M \otimes_A -$ to the above sequence, we get

$$0 \to \mathrm{Tor}_1^A(M, L_{i-1}) \to M \otimes_A L_i \to M \otimes_A X \to M \otimes_A L_{i-1} \to 0 \quad (\text{ex}).$$

Note that the isomorphism $\mathrm{Tor}_1^A(M, L_{i-1}) \cong \mathrm{Tor}_i^A(M,k)$ is derived from Lemma 2.2 and the following exact seqeunce:

$$0 \to L_{i-1} \to F^*_{d-i+2} \otimes_A K_A \to \cdots \to F^*_{d-1} \otimes_A K_A \to X_k \to k \to 0 \quad (\text{ex}).$$

Now let J be any parameter ideal for $M \otimes_A L_i$. Then as J is also one for M, we get

$$0 \to \mathrm{Tor}_1^A(M/JM, L_{i-1}) \to \frac{M \otimes L_i}{J(M \otimes L_i)} \to \frac{M \otimes X}{J(M \otimes X)} \to \frac{M \otimes L_{i-1}}{J(M \otimes L_{i-1})} \to 0$$

and thus

$$\begin{aligned} &\ell\left(\frac{M \otimes L_i}{J(M \otimes L_i)}\right) - e_J(M \otimes_A L_i) \\ &= \ell(\mathrm{Tor}_1^A(M/JM, L_{i-1})) + I_A(M \otimes_A X) - I_A(M \otimes_A L_{i-1}) \\ &= \ell(\mathrm{Tor}_i^A(M/JM, k)) - I_A(M \otimes_A L_{i-1}). \end{aligned}$$

Hence $M \otimes_A L_i$ is a Buchsbaum A-module of dimension r.

Furthermore, from the similar argument as in Step 2, we can easily get as follows:

$$H^0_{\mathfrak{m}}(M \otimes_A L_i) \cong \mathrm{Tor}_i^A(M,k)$$
$$H^j_{\mathfrak{m}}(M \otimes_A L_i) \cong H^{j-1}_{\mathfrak{m}}(M \otimes_A L_{i-1}) \cong \mathrm{Tor}^A_{i-j}(M,k)$$

for all $1 \leq j < r$.

Finally, noting that $\operatorname{Tor}^A_{d-j}(M,k) = 0$ whenever $j < r = d - \operatorname{pd}_A M$, we can get the required assertion. □

2.3. The general case

Throughout of this subsection, assume that A is a CM local ring of dimension d, M is a perfect A-module and N is a maximal surjective Buchsbaum A-module.

We now recall the notion of L-basis. See [10, Chapter 1] for detail.

Let L be a finite A-module and $I \subseteq A$ an ideal. Suppose $\dim L/IL = 0$. Let $r = \dim L$. A system of elements $a_1, \ldots, a_t$ of A is called an L-basis of I if the following conditions are fulfilled;

(i) $a_1, \ldots, a_t$ is a minimal basis of I.

(ii) For every system $i_1, \ldots, i_r$ of integers with $1 \leq i_1 < \cdots < i_r \leq t$ the elements $a_{i_1}, \ldots, a_{i_r}$ form an s.o.p. of L.

As is known an L-basis of I always exists. Furthermore, the following criterion for Buchsbaum modules is known ([11, Proposition 3.2]).

L is a Buchsbaum A-module if and only if L is F.L.C. and there exists an L-basis $a_1, \ldots, a_v$ of $\mathfrak{m}$ that satisfies the following condition: For any $1 \leq i_1 < \cdots < i_r \leq v$, one has the equality

$$\ell(L/JL) - e_J(L) = \sum_{i=0}^{r-1} \binom{r-1}{i} \ell(H^i_{\mathfrak{m}}(L)),$$

where $J = (a_{i_1}, \ldots, a_{i_r})A$ and $r = \dim L$.

We are now ready to prove Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1. We may assume that A is complete and $\dim M = r \geq 1$. Let N be a maximal surjective Buchsbaum A-module and

$$0 \to N \to Y \to X \to 0 \quad \text{(ex)}$$

be its finite injective hull over A, that is, Y is a finite A-module of finite injective dimension and X is a maximal CM A-module (cf. [1]).

Since X is a maximal CM A-module, Y is also a maximal surjective Buchsbaum A-module with $\operatorname{depth} Y = \operatorname{depth} N$. Moreover, as $\operatorname{Tor}^A_1(M,X) = 0$, we get

$$0 \to M \otimes_A N \to M \otimes_A Y \to M \otimes_A X \to 0 \quad \text{(ex)}. \tag{1}$$

From Propositions 2.7 and 2.4, we have that $M \otimes_A Y$ is a Buchsbaum A-module and $M \otimes_A X$ is a CM A-module with $\dim M \otimes_A N = \dim M \otimes_A Y = \dim M \otimes_A X = r$. Moreover, we get

$$\begin{aligned}\operatorname{depth} M \otimes_A N &= \operatorname{depth} M \otimes_A Y \\ &= \max\{\operatorname{depth} Y - \operatorname{pd}_A M, 0\} \\ &= \max\{\operatorname{depth} N - \operatorname{pd}_A M, 0\}.\end{aligned}$$

Let $a_1,\dots,a_v$ be an M-basis of $\mathfrak{m}$. Noting that this basis is also an $M \otimes_A N$-basis of $\mathfrak{m}$, in order to prove that $M \otimes_A N$ is a Buchsbaum A-module, it suffices to show that the following formula holds: For any $1 \le i_1 < \cdots < i_r \le v$,

$$\ell\left(\frac{M \otimes N}{J(M \otimes N)}\right) - e_J(M \otimes_A N) = \sum_{i=0}^{r-1} \binom{r-1}{i} \ell\left(H^i_{\mathfrak{m}}(M \otimes_A N)\right),$$

where $J = (a_{i_1},\dots,a_{i_r})A$.

Now suppose that J is a parameter ideal for M. As $\mathrm{Tor}_1^A(A/J, M \otimes_A X) = 0$, we get

$$\begin{aligned}
&\ell\left(\frac{M \otimes N}{J(M \otimes N)}\right) - e_J(M \otimes_A N)\\
&= \ell\left(\frac{M \otimes Y}{J(M \otimes Y)}\right) - e_J(M \otimes_A Y) - \left\{\ell\left(\frac{M \otimes X}{J(M \otimes X)}\right) - e_J(M \otimes_A X)\right\}\\
&= \sum_{i=0}^{r-1} \binom{r-1}{i} \ell(H^i_{\mathfrak{m}}(M \otimes_A Y))\\
&= \sum_{i=0}^{r-1} \binom{r-1}{i} \ell(H^i_{\mathfrak{m}}(M \otimes_A N))
\end{aligned}$$

and thus $M \otimes_A N$ is a Buchsbaum A-module.

In (4), in addition, suppose $\operatorname{depth} N \ge \operatorname{pd}_A M = s$. From (1), we have that $M \otimes_A N$ is a surjective Buchsbaum A-module if and only if so is $M \otimes_A Y$.

Thus we may assume that $\operatorname{id}_A N < \infty$. Then as

$$M \otimes_A N \cong \bigoplus_{t=s}^{d} (M \otimes_A L_t)^{h^t_A(N)},$$

it suffices to show that $M \otimes_A L_t$ is a surjective Buchsbaum A-module for $t = s,\dots,d$.

From Lemma 2.2, we get

$$\mathrm{Tor}_i^A(M, L_t) \cong \mathrm{Tor}_{i+t}^A(M, k) = 0 \quad \text{for all } i \ge 1, t \ge s.$$

From Lemma 2.3 and [12, Theorem 1.2], we get

$$\begin{aligned}
\mu_A^n(M \otimes_A L_t) &= \sum_{p=0}^{s} \beta_p^A(M)\, \mu_A^{n+p}(L_t)\\
&= \sum_{p=0}^{s} \beta_p^A(M)\, \beta_{n+p-t}^A(k)
\end{aligned}$$

for all $n < r = d - s$.

On the other hand, from Proposition 2.7, we have

$$\begin{aligned}\sum_{j=0}^{n}\beta_{n-j}^{A}(k)\,h_{A}^{j}(M\otimes_{A}L_{t}) &= \sum_{j=0}^{n}\beta_{n-j}^{A}(k)\,\beta_{t-j}^{A}(M)\\ &= \sum_{p=t-n}^{s}\beta_{p}^{A}(M)\,\beta_{n+p-t}^{A}(k)\\ &= \sum_{p=0}^{s}\beta_{p}^{A}(M)\,\beta_{n+p-t}^{A}(k)\end{aligned}$$

for all $n < r = d - s$. Therefore we conclude that $M \otimes_A L_t$ is a surjective Buchsbaum A-module for all $t \geq s$ ([12, Theorem (1.2)]). Hence so is $M \otimes_A N$. □

Proof of Corollary 1.2. We need only to prove the implication (2) ⇒ (1). Suppose that $\dim M \geq 1$ and $M \otimes_A N$ is a CM A-module. We may assume that A is complete. Let

$$0 \to N \to Y \to X \to 0 \quad \text{(ex)}$$

be a finite injective hull of N over A. Then Y is also a maximal surjective Buchsbaum A-module. As $M \otimes_A X$ is a CM A-module with $\dim M \otimes_A X = r$, we have that $M \otimes_A N$ is CM if and only if so is $M \otimes_A Y$ (cf. Eq. (1)).

Now suppose that L_t is a direct summand of Y for some t. Then since $M \otimes_A L_t$ is also a direct summand of $M \otimes_A Y$, $M \otimes_A L_t$ is CM. Thus from Proposition 2.7 we get

$$\operatorname{Tor}_{t-j}^{A}(M,k) = 0 \quad \text{for all } j = 0, 1, \ldots, r-1.$$

It follows that $t - (r - 1) \geq \operatorname{pd}_A M + 1 = d - r + 1$, that is, $t = d$. Hence Y is a maximal CM A-module, and thus so is N. □

Conjecture 2.10. *Let M be a perfect A-module and N a maximal surjective Buchsbaum A-module. Then $M \otimes_A N$ is a surjective Buchsbaum A-module.*

Conjecture 2.11. *Let M be a perfect A-module of positive dimension and N a finite A-module with $\dim N = \dim A$. If $M \otimes_A N$ is a CM A-module, then N is a maximal CM A-module.*

3. Examples

We first recall the following remark [8].

Let A be a local ring. Then

(1) $\operatorname{syz}_t^A(k) = 0$ if and only if A is regular and $t \geq \dim A + 1$.

(2) If $\operatorname{syz}_t^A(k) \neq 0$, then $\operatorname{Supp}_A(\operatorname{syz}_t^A(k)) = \operatorname{Spec}(A)$.

Using the similar argument as in the proof of Proposition 2.7, one can get as follows:

Proposition 3.1. *Let A be a local ring and $t \geq 1$ an integer. Suppose that M is a CM A-module. If $\operatorname{syz}_t^A(k) \neq 0$, then $M \otimes_A \operatorname{syz}_t^A(k)$ is a Buchsbaum A-module with $\dim M \otimes_A \operatorname{syz}_t^A(k) = \dim M$.*

Furthermore, we have

$$H_{\mathfrak{m}}^j(M \otimes_A \operatorname{syz}_t^A(k)) \cong \operatorname{Tor}_{t-j}^A(M,k)$$

for all $j < \dim M$.

From this proposition, we get two corollaries as follows:

Corollary 3.2. *Let A be a regular local ring and M a perfect A-module. If N is a maximal Buchsbaum A-module, then $M \otimes_A N$ is a Buchsbaum A-module with $\dim M \otimes_A N = \dim M$.*

Proof. The statement follows from Proposition 3.1 [10, Chapter 1, Corollary 2.16; 3]. □

Corollary 3.3. *Let A be a local ring and M a CM A-module with $\dim M \geq 1$. If $M \otimes_A \operatorname{syz}_t^A(k)$ is a CM A-module for some positive integer t, then M is perfect.*

Proof. By the assumption, one can easily get

$$0 = H_{\mathfrak{m}}^0(M \otimes_A \operatorname{syz}_t^A(k)) \cong \operatorname{Tor}_t^A(M,k).$$

Thus we have $\operatorname{pd}_A M < \infty$. Hence A is a CM local ring and M is perfect. (cf. 1.8 and 1.9). □

Let A be a CM local ring, M a perfect A-module and N a finite A-module with $\dim N = \dim A$. Then the condition of Theorem 1.1(1) does not imply that N is a surjective Buchsbaum A-module. See below.

Example 3.4 (*e.g. Stückrad and Vogel* [10, *Chapter* 1, *Section* 2, *Example* 2.18]). Let $A = k[[X, Y, Z, W]]/(X^2Z, YW)$ and

$$N = A/(X^2W, YZ)A \cong k[[X, Y, Z, W]]/(X^2, Y) \cap (Z, W).$$

Put $a = x + z$. Then

(1) a is an A-regular element. In particular, A/aA is perfect.
(2) N is F.L.C. with $\dim N = \dim A = 2$.
(3) N is not a Buchsbaum A-module.
(4) $N/aN \cong A/aA \otimes_A N$ is a surjective Buchsbaum A-module.

In the rest of this section, we consider the following question.

Question 3.5. Let M be a surjective Buchsbaum A-module of finite projective dimension and N a maximal surjective Buchsbaum A-module. Then when is $M \otimes_A N$ surjective Buchsbaum A-module ?

For this question, we give two answers as follows.

Proposition 3.6 (cf. Kawasaki [5, Theorem 3.3(iii)]). *Let A be a CM local ring with canonical module K_A. Let M be a finite A-module of finite projective dimension and N a maximal CM A-module.*

If $M \otimes_A K_A$ is a surjective Buchsbaum A-module, then so is $M \otimes_A N$.

Let $\mathbb{H}.$ (resp. $\mathbb{G}.$) be a minimal free resolution of M (resp. $\mathrm{Hom}_A(N, K_A)$) over A. For any complex $\mathbb{X}.$, let $\mathbb{X}.(n)$ denote the shifting of $\mathbb{X}.$ in degree n. Then $\mathrm{Hom}_A(\mathbb{H}., A)(-d) \otimes_A \mathbb{G}.$ gives a minimal free resolution of $D(M \otimes_A N) = \mathrm{Hom}_A(M \otimes_A N, D_A)$, where D_A denotes a normalized dualizing complex of A; see the proof of [5, Theorems 3.1 and 3.3]. From here, one can get the proof of the above proposition by the same argument as in the proof of [5, Theorem 3.3].

On the other hand, the following example gives a negative answer for Question 3.5.

Example 3.7. Let A be a Gorenstein local ring with $d = \dim A \geq 2$ and let M, N be maximal surjective Buchsabaum A-modules of finite projective dimension. Put $t = \mathrm{depth}\, M$ and $u = \mathrm{depth}\, N$. Suppose $t + u \geq d$ and $t, u \leq d - 1$. Then $M \otimes_A N$ is not a surjective Buchsbaum A-module.

Proof. We may assume that A is complete, $M = L_t$ and $N = L_u$; see Theorem 2.8. Moreover, assume that $t + u \geq d, t \leq u \leq d - 1$.

Suppose that $L_t \otimes_A L_u$ is a surjective Buchsbaum A-module. By Lemma 2.2, we have $\mathrm{Tor}_i^A(L_t, L_u) = 0$ for all $i \geq 1$. Put $\theta = t + u - d$. Then from Lemma 2.3, we get $\mathrm{depth}\, L_t \otimes_A L_u = \theta$ and for all $\theta \leq i \leq t$,

$$\begin{aligned}\mu_A^i(L_t \otimes_A L_u) &= \sum_{p=0}^{d-t} \beta_p^A(L_t) \cdot \mu_A^{i+p}(L_u) \\ &= \sum_{p=u-i}^{d-t} \beta_p^A(L_t) \cdot \beta_{p+i-u}^A(k) \\ &= \sum_{p=u-i}^{d-t} \beta_{d-t-p}^A(k) \cdot \beta_{p+i-u}^A(k).\end{aligned}$$

Thus

$$\mu_A^{i+\theta}(L_t \otimes_A L_u) = \sum_{q=0}^{i} \beta_{i-q}^A(k) \cdot \beta_q^A(k) \tag{2}$$

for all $0 \leq i \leq t - \theta$.

On the other hand, by [12, Theorem 1.2], we get

$$\mu_A^{i+\theta}(L_t \otimes_A L_u) = \sum_{q=0}^{i} \beta_{i-q}^A(k) \cdot h_A^{q+\theta}(L_t \otimes_A L_u) \tag{3}$$

for all $0 \leq i \leq t - \theta$. Thus from Eqs. (2) and (3), we have $h_A^j(L_t \otimes_A L_u) = \beta_{j-\theta}^A(k)$ for all $0 \leq j \leq t$.

Furthermore, from the hypothesis, we can write as

$$L_t \otimes_A L_u \cong \bigoplus_{j=0}^{d} L_j^{h_A^j(L_t \otimes L_u)}.$$

Therefore we get

$$\begin{aligned} \beta_{d-t}^A(k) \cdot \beta_{d-u}^A(k) &= \mu_A(L_t \otimes_A L_u) \\ &= \sum_{j=0}^{d} \mu_A(L_j) \cdot h_A^j(L_t \otimes_A L_u) \\ &\geq \sum_{j=0}^{t} \beta_{d-j}^A(k) \cdot \beta_{j-0}^A(k) \\ &> \beta_{d-t}^A(k) \cdot \beta_{d-u}^A(k) \end{aligned}$$

This is a contradiction and completes the proof of this example. □

4. Serre conditions

As is known if a local ring A admits a CM A-module of finite projective dimension then A itself is a CM local ring.

We can improve this result as follows:

Proposition 4.1. *Let A be a local ring and n a nonnegative integer. If A admits a finite A-module M of finite projective dimension which satisfies S_n, then A itself satisfies S_n.*

$$M : S_n \iff \operatorname{depth} M_P \geq \min(n, \dim M_P) \quad \textit{for every } P \in \operatorname{Supp}_A(M).$$

Proof. Fix $P \in \operatorname{Spec}(A)$. We must show $\operatorname{depth} A_P \geq \min(n, \dim A_P)$.

Case 1: When $P \in \operatorname{Supp}_A(M)$, we have

$$\operatorname{depth} A_P = \operatorname{depth} M_P + \operatorname{pd}_{A_P} M_P.$$

When $\dim M_P \geq n$, one has

$$\operatorname{depth} A_P \geq \operatorname{depth} M_P \geq \min(n, \dim M_P) = n.$$

Otherwise, then M_P is a CM A_P-module and so that A_P is CM. Thus we get the required inequality.

Case 2: When $P \notin \operatorname{Supp}_A(M)$, we put $I = \operatorname{ann}_A(M)$. If neessary, localizing at a prime ideal $Q \in \operatorname{Min}(A/I + P)$, we may assume that $\mathfrak{m} = \sqrt{I + P}$.

By Intersection Theorem, we have

$$\dim A/P \leq \operatorname{pd}_A M + \dim(A/P \otimes_A M) = \operatorname{pd}_A M,$$

and so

$$\begin{aligned}\operatorname{depth} A_P &\geq \operatorname{grade}_A A/P \\ &\geq \operatorname{depth} A - \dim A/P \\ &\geq \operatorname{depth} A - \operatorname{pd}_A M \\ &= \operatorname{depth} M.\end{aligned}$$

When $\dim M \geq n$, $\operatorname{depth} A_P \geq \operatorname{depth} M \geq n$. Otherwise, then M is CM, and so that A is CM. Hence $\operatorname{depth} A_P \geq \min(n, \dim A_P)$. □

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